Journal of Applied Mechanics and Technical Physics, Vol. 35, No. 1, 1994

SMALL-DIAMETER STRUCTURE WITH UNILATERAL INSIDE CONTACTS

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UDC 539.3

An asymptotic expansion was formally constructed in [1] for the problem of contacting elastic bodies with a small diameter ε. Passage to the limit (at $\varepsilon \to 0$) yields a unidimensional problem. The corresponding equilibrium equations coincide with the classical equations [2, 3]. The constitutive equations are obtained by solving a so-called mesh problem on a mesh with a periodic structure. This approach can be used in particular to study textiles [4].

1. Formulation of the Problem. We will examine a periodic structure formed by elastic elements that can be placed in ideal contact (Fig. 1). The structure occupies the region Ω_ε, having the characteristic diameter $\varepsilon \ll 1$ (which is formalized in the form $\varepsilon \to 0$ [1, 2]). A cell of the structure (shown in Fig. 2 in terms of the dimensionless variables $y = x/\varepsilon$) has the same characteristic dimension. The elastic constants of the elements of this structure will be designated as $\varepsilon^{-4}a_{ijkl}(x/\varepsilon)$. The functions a_{ijkl} are assigned to be periodic with respect to y_1. They have the period m (Fig. 2) and are bounded. The coefficient ε^{-4} is introduced to account for the order of bending stiffness. The presence of contacts is accounted for as follows [5, 6]. We introduce the function space

$$V_\varepsilon = \{u \in H^1(\Omega_\varepsilon):\ u(x) = 0 \text{ on } \Gamma_\varepsilon^0 \text{ (see Fig.1)}\}$$

and formulate the condition for the displacements u^ε of elements of the structure:

$$u^\varepsilon \in M(V_\varepsilon) = \{u \in V_\varepsilon:\ [un] = 0 \text{ on the contact surfaces }\} \tag{1.1}$$

(where **n** is a normal vector). By contact surfaces, we mean surfaces on which the structural elements might come into contact. If we *a priori* exclude the possibility of the existence of such surfaces, then the entire surface of the elements is taken as the contact surface. In local variables y within a given cell $P_1 = \varepsilon^{-1}P_\varepsilon = \{y = x/\varepsilon:\ x \in P_\varepsilon\}$ (see Figs. 1 and 2) the contact condition has the form

$$u^\varepsilon(x_1, y) \in \tilde{M} = \{u \in H^1(P_1):\ [un] = 0 \tag{1.2}$$

on the contact surfaces and **u** is periodic with respect to y_1, having the period m}. The variable x_1 in (1.2) is "frozen." The braces in (1.1) and (1.2) denote a jump — the difference in the values of the function on different sides of a contact surface [6].

The displacements u^ε are found by solving the variational inequality [5, 6]

$$\int_{\Omega_\varepsilon} \sigma_{ij}^\varepsilon (u^\varepsilon - v)_{i,j} dv - \varepsilon^{-2} \int_{\Gamma_\varepsilon} g(u^\varepsilon - v)\, ds \geq -\varepsilon^{-2} \int_{\Omega_\varepsilon} f(u^\varepsilon - v)\, dv \tag{1.3}$$

for any $v \in M(V_\varepsilon)$, where

$$\sigma_{ij}^\varepsilon = \varepsilon^{-4} a_{ijkl}(x/\varepsilon)\, u_{k,l}^\varepsilon. \tag{1.4}$$

Summation is carried out over the repeating indices.

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 1, pp. 128-135, January-February, 1994. Original article submitted November 25, 1992; revision submitted February 9, 1993.

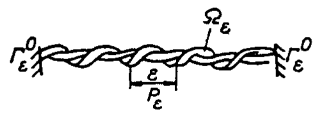

Fig. 1

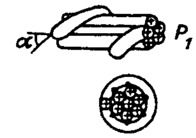

Fig. 2

Note 1. The multiplier ε^{-4} in Hooke's law (1.4) accounts for the fact that the bending stiffness of the beam is proportional to its diameter to the fourth power [7]. The proportionality of the forces **f** and **g** to the quantity ε^{-2} is related to allowance for their orders in beam models [7, 3].

If the conditions a_{ijkl}, **f**, **g** $\in C(R^3)$ are satisfied and the region Ω_ε has the boundary C^1, then problem (1.1), (1.3), (1.4) is unambiguously solvable in $M(V_\varepsilon)$ for any $\varepsilon > 0$ [5, 6]. Our goal is to analyze the problem for $\varepsilon \to 0$.

2. Formal Asymptotic Expansion. We will analyze the problem by using the two-scale method [6], which is based on simultaneous use of the initial **x** and local $\mathbf{y} = \mathbf{x}/\varepsilon$ variables and representation of the solution in the form of a series in powers of ε, i.e., we will construct a formal asymptotic expansion [1] in the form proposed in [3]

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(x_1) + \varepsilon \mathbf{u}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(x_1, \mathbf{y}); \tag{2.1}$$

$$\mathbf{v} = \mathbf{v}^{(0)}(x_1) + \varepsilon \mathbf{v}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}^{(k)}(x_1, \mathbf{y}); \tag{2.2}$$

$$\sigma_{ij}^\varepsilon = \varepsilon^{-4}\sigma_{ij}^{(-4)} + \ldots = \sum_{m=-4}^{\infty} \varepsilon^m \sigma_{ij}^{(m)}(x_1, \mathbf{y}), \tag{2.3}$$

where the functions in the right sides of (2.1)-(2.3) are periodic with respect to y_1, having the period m.

Let us change over to the variables x_1, **y** in (1.1) and (1.3), (1.4) as well. In these variables, the functions $f(x_1, \mathbf{y})$ are differentiated according to the rule

$$\frac{\partial}{\partial x_1} \to \frac{\partial}{\partial x_1} + \varepsilon^{-1}\frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial x_\alpha} \to \varepsilon^{-1}\frac{\partial}{\partial y_\alpha} \quad (\alpha = 2, 3). \tag{2.4}$$

Here and below, the Greek-letter indices take values of 2 and 3, the Roman-letter indices have values of 1, 2, and 3, and the following notation is used: $,1x = \partial/\partial x_1$ and $,iy = \partial/\partial y_i$.

Substitution of the variables in the integrals from (1.3) and substitution of (2.4) yields

$$\sum_{k=0}^{\infty}\sum_{m=-4}^{\infty} \varepsilon^2 \int_{\Omega_1} (\varepsilon^{m+k-1}\sigma_{ij}^{(m)} w_{i,jy}^{(k)} + \varepsilon^{m+k}\sigma_{i1}^{(m)} w_{i,1x}^{(k)})\, dv + \\ + \sum_{k=0}^{\infty} \varepsilon^k \int_{\Gamma_1} \mathbf{g}\mathbf{w}^{(k)} ds \geq - \sum_{k=0}^{\infty} \varepsilon^k \int_{\Omega_1} \mathbf{f}\mathbf{w}^{(k)} dv \tag{2.5}$$

for any $\mathbf{v} \in M(\Omega_1)$.

Here, $\Omega_1 = \{(x_1, y_2, y_3): \mathbf{x} \in \Omega_\varepsilon\}$, $\Gamma_1 = \{(x_1, y_2, y_3): \mathbf{x} \in \Gamma_\varepsilon\}$ have a characteristic diameter equal to unity; $\mathbf{w}^{(k)} = \mathbf{u}^{(k)} - \mathbf{v}^{(k)}$.

Insertion of (2.1) and (2.3) into Hooke's law (1.4) gives [3]

$$\sigma_{ij}^{(m)} = a_{ijkl}(\mathbf{y})\, u_{k,ly}^{(m+5)} + a_{ijk1}(\mathbf{y})\, u_{k,1x}^{(m+4)} \quad (m = -4, -3, \ldots). \tag{2.6}$$

3. Equilibrium Equations. We will only briefly discuss the derivation of the equilibrium equations for thin structures, since this procedure has by now become standard [2, 3, 8]. We will examine Eq. (2.5) with $\mathbf{w} = \mathbf{w}^0(x_1) \in C^1([-1, 1])$. In the given case, it takes the form

$$\int_{\Omega_1} \varepsilon^{m+1}\sigma_{i1}^{(m)} w_{i,1x}^{(0)} dv - \int_{\Gamma_1} g w^{(0)} ds \geq - \int_{\Omega_1} f w^{(0)} dv \quad (m = -4, -3, \ldots) \tag{3.1}$$

for any $v^{(0)} \in C^1([-1, 1])$.

Note 2. The following relationship exists between the integrals of rapidly oscillating functions of the form $f(x_1, x/\varepsilon)$ and their means [3] at $\varepsilon \to 0$

$$\int_{\Omega_1} f(x_1, x/\varepsilon)\, dv \to \int_{-1}^{1} \langle f \rangle (x_1)\, dx_1,$$
$$\int_{\Gamma_1} f(x_1, x/\varepsilon)\, ds \to \int_{-1}^{1} \langle f \rangle_\gamma (x_1)\, dx_1,$$

where $\langle f \rangle = m^{-1} \int_{P_1} f(x_1, y)\, dy$ is the mean over a cell; $\langle f \rangle_\gamma = m^{-1} \int_\gamma f(x_1, y)\, dy$ is the mean over the lateral surface of the cell. It should be noted that we are examining a case in which the projections of Ω_1 (and Ω_ε) on the Ox_1 axis form the segment $[-1, 1]$ (see Fig. 1).

With allowance for Note 2, we can use (3.1) at $\varepsilon \to 0$ (keeping in mind that $v^{(0)} \in C([-1, 1])$ to obtain the equalities

$$\langle \sigma_{i1}^{(m)} \rangle_{,1x} = 0 \quad (m = -4, -3),$$
$$\langle \sigma_{i1}^{(-2)} \rangle_{,1x} = \langle g_i \rangle_\gamma + \langle f_i \rangle. \tag{3.2}$$

The quantities $\langle \sigma_{i1}^{(m)} \rangle$ represent axial forces.

Now, in (2.5) we put $w = \varepsilon w^{(1)}(x_1, y) = \varepsilon(y_2 v_2(x_1) + y_3 v_3(x_1))$ (i.e., $v = u^{(1)} + \varepsilon w^{(1)}$. Here, $v \in M(\Omega_1)$, which can be proven directly). For the case being examined, (2.5) takes the form

$$\sum_{m=-4}^{\infty} \varepsilon^2 \int_{\Omega_1} [\varepsilon^m \sigma_{ij}^{(m)} (v_{2i}\delta_{2j} + v_{3i}\delta_{3j}) + \varepsilon^{(m+1)} \sigma_{i1}^{(m)} (y_2 v_{2i,1x} + y_3 v_{3i,1x})]\, dv +$$
$$+ \int_{\Gamma_1} g (y_2 v_2 + y_3 v_3)\, ds \geq - \sum_{m=-4}^{\infty} \int_{\Omega_1} f (y_2 v_2 + y_3 v_3)\, dv.$$

Introducing moments by means of the formula [3] $M_{ij} = \langle y_j \sigma_{i1}^{(m)} \rangle$ and allowing for Note 2 and the fact that $v_2, v_3 \in C^1([-1, 1])$, we find

$$\langle \sigma_{i\alpha}^{(-4)} \rangle = 0 \quad (\alpha = 2, 3,\ i = 1, 2, 3),$$
$$-M_{\alpha i, 1x}^{(-4)} + \langle \sigma_{i\alpha}^{(-3)} \rangle = 0,$$
$$-M_{\alpha i, 1x}^{(-3)} + \langle \sigma_{i\alpha}^{(-2)} \rangle = \langle g_i y_\alpha \rangle_\gamma + \langle f_i y_\alpha \rangle. \tag{3.3}$$

4. Constitutive Relations. The specifics of the given problem are manifest to the greatest extent when the constitutive relations are derived [1-3, 6-12]. Let us proceed to the construction of these relations.

We choose a test function in the form $v = \varepsilon v^{(1)}(y)$, where $v^{(1)}(y)$ is periodic with respect to y_1 and has the period m. For such a choice, by equating terms of order ε^{-4} in (2.5) we have

$$\int_{\Omega_1} \sigma_{ij}^{(-4)} w_{i,jy}^{(1)} dv \geq 0 \quad (w^{(1)} = u^{(1)} - v^{(1)})$$

for any $v \in M(\Omega_1)$. Considering the periodicity of the functions with respect to y_1 and taking Note 2 into account, we can rewrite this inequality in the below form (see the proof in [6])

$$\int_{P_1} \sigma_{ij}^{(-4)} w_{i,jy}^{(1)} dy \geqslant 0 \tag{4.1}$$

for any $v^{(1)} \in \tilde{M}$, where in accordance with (2.6)

$$\sigma_{ij}^{(-4)} = a_{ijkl}(y)\, u_{k,ly}^{(1)} + a_{ijp1}(y)\, u_{p,1x}^{(0)}(x_1). \tag{4.2}$$

In the analogous linear problem (in the absence of contacts), an important role is played by the fact that it is possible to obtain a solution corresponding to the term $a_{ij\alpha 1}(y)\, u_{\alpha,1x}^{(0)}(x_1)$ in (4.2), as well as a solution of the homogeneous problem in explicit form [3]. These solutions are also useful to have in the present case, if they are obtained by another method. Using the approach in [9], we represent the solution of (4.1), (4.2) as

$$u^{(1)} = U^{(1)} - y_\alpha u_{\alpha,1x}^{(0)}(x_1)\, e_1 - y_\alpha s_{\bar{\alpha}} \varphi(x_1)\, e_{\bar{\alpha}} + V(x_1), \tag{4.3}$$

where $\bar{\alpha} = \begin{cases} 3 \text{ at } \alpha = 2, \\ 2 \text{ at } \alpha = 3; \end{cases}$ $s_1 = 0$; $s_2 = 1$; $s_3 = -1$; e_i are orthonormal basis vectors of the standard coordinate system; $\varphi(x_1)$, $V(x_1)$ are functions (the determination of which is a subject for further study) satisfying the condition $\varphi(\pm 1) = V(\pm 1) = 0$; $U^{(1)}$ is a new unknown function; the remaining terms in the right side of (4.3) are introduced as in [3]. Insertion of (4.3) into (4.1) gives

$$\int_{P_1} \sigma_{ij}^{(-4)} (U^{(1)} - v^{(1)})_{i,jy} dv - \langle \sigma_{\alpha 1}^{(-4)} \rangle\, u_{\alpha,1x}^{(0)} - \langle \sigma_{23}^{(-4)} - \sigma_{32}^{(-4)} \rangle\, \varphi(x_1) \geqslant 0.$$

The last two terms in this inequality are zeros; the first is zero due to (3.5); the second is zero due to the symmetry of σ_{ij} with respect to ij (and the consequent symmetry of $\sigma_{ij}^{(m)}$, $m = -4, -3, \ldots$). Also, insertion of (4.3) into (4.2) yields

$$\sigma_{ij}^{(-4)} = a_{ijkl}(y)\, U_{k,ly}^{(1)} + a_{ij11}(y)\, u_{1,1x}^{(0)}(x_1). \tag{4.4}$$

As a result, we arrive at the problem

$$\int_{P_1} [a_{ykl}(y)\, U_{k,ly}^{(1)} + a_{ij11}(y)\, u_{1,1x}^{(0)}(x_1)]\, (U^{(1)} - v^{(1)})_{i,jy} dy \geqslant 0 \tag{4.5}$$

for any $v \in \tilde{M}$.

Note 3. We introduce the function $W = U^{(1)} + u_{1,1x}^{(0)} y_1 e_1$ and observe that (4.4) represents the problem of the microscopic deformation of a cell (of a so-called cellular structure [13-15]) corresponding to averaged (macroscopic) strains $u_{1,1x}^{(0)}$. This observation is useful when a mesh problem is being solved by modeling it by simplified structures [4] or when allowance is being made for its specific features (such as in [10, 13-17]).

Variational inequality (4.5) is analogous to that obtained in [6] in the averaging of a monolithic body with a periodic system of distributed cracks. The difference in [6] was that the periodicity condition was formulated for all of the variables and the cell had no free surface. However, these differences are unimportant for proving the solvability of problem (4.5) and the uniqueness of its solution to within the functions φ, V (the proof is analogous to that given in [6]). We will use $U(u_{1,1x}^{(0)})$ to represent the solution of (4.5). Having inserted it into (4.4) and having averaged the result, we obtain the constitutive equation

$$\langle \sigma_{11}^{(-4)} \rangle = \langle a_{ijkl}(y)\, U_{k,ly}(u_{1,1x}^{(0)}) + a_{ij11}(y)\, u_{1,1x}^{(0)} \rangle, \tag{4.6}$$

which connects the axial force with the axial strain $u_{1,1x}^{(0)}$. Following [6], we establish that 1) (4.6) is the hyperelastic law; 2) with $i\alpha = 11$ and with the boundary condition $u_1^{(0)}(\pm 1) = 0$ [this condition following from initial boundary condition (1.1) and expansion (2.1)], Eq. (3.3) has the unique solution $u_1^{(0)}(x_1) = 0$. From this, in turn, we find that $U_1^{(0)}(y) = 0$ and (4.3) takes the form

$$u^{(1)} = -y_\alpha u^{(0)}_{\alpha,1x}(x_1)\, e_1 - y_\alpha s_{\bar\alpha}\varphi(x_1)\, e_{\bar\alpha} + V(x_1). \tag{4.7}$$

Then $\sigma_{ij}^{(-4)} = 0$ [see (4.2)], and since $M_{ij}^{(-4)} = 0$ we obtain $\langle\sigma_{i\alpha}^{(-3)}\rangle = 0$ [see (3.3)].

Let us examine the choice of test function $v = \varepsilon v^{(1)}(y)$. As above, the terms from (2.5) are of the order ε^{-3}. For these terms we have

$$\int_{P_1} \sigma_{ij}^{(-3)} w^{(1)}_{i,jy} dy \geqslant 0 \quad (w^{(1)} = u^{(1)} - v^{(1)}) \tag{4.8}$$

for any $v \in \tilde{M}$, where in accordance with (2.6)

$$\sigma_{ij}^{(-3)} = a_{ijkl}(y)\, u^{(2)}_{k,ly} + a_{ijk1}(y)\, u^{(1)}_{k,1x}. \tag{4.9}$$

Substitution of (4.7) into (4.9) gives

$$\begin{aligned}\sigma_{ij}^{(-3)} = a_{ijkl}(y)\, u^{(2)}_{k,ly} + a_{ijk1}(y)\, V_{k,1x}(x_1) - \\ a_{ij11}(y)\, y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ij\bar\alpha 1}(y)\, y_\alpha s_\alpha \varphi_{,1x}(x_1).\end{aligned} \tag{4.10}$$

Equation (4.10) includes the quantities $V_{k,1x}$. Of the latter, the only quantity having physical significance is $V_{1,1x}$ — corresponding to axial deformation. In the linear case, if we use the solutions obtained in explicit form for the mesh problem (4.1), (4.2) we can exclude terms containing $V_{\alpha,1x}$ ($\alpha = 2, 3$) from (4.10) [3]. In the present case, as in [9] we represent $u^{(2)}$ in the form

$$u^{(2)} = U^{(2)} - y_\alpha V_{\alpha,1x}(x_1)\, e_1. \tag{4.11}$$

Inserting (4.11) into (4.8), with allowance for Note 2 we find

$$\int_{\Omega_1} \sigma_{ij}^{(-3)} (U^{(2)} - v^{(2)})_{i,jy} dv - \int_{-1}^{1} \langle\sigma_{1\alpha}^{(-3)}\rangle V_{\alpha,1x} dx_1 \geqslant 0. \tag{4.12}$$

The last integral in (4.12) is equal to

$$\int_{-1}^{1} \langle\sigma_{1\alpha}^{(-3)}\rangle_{,1x} V_\alpha(x_1)\, dx_1 + \langle\sigma_{1\alpha}^{(-3)}\rangle V_\alpha(x_1)\big|_{x_1=-1}^{x_2=1}.$$

When $m = -3$, the last equality is a consequence of (3.2) and the fact that $V_\alpha(\pm 1) = 0$. Then taking into account the periodicity of the function in the remaining integral in (4.12), we arrive (as in [6]) at the following inequality for a cell of the structure

$$\int_{P_1} \sigma_{ij}^{(-3)} (U^{(2)} - v^{(2)})_{i,jy} dy \geqslant 0, \tag{4.13}$$

which with allowance for (4.10) and (4.11) takes the form

$$\begin{aligned}\int_{P_1} [a_{ijkl}(y)\, U^{(2)}_{k,ly} - a_{ij\beta 1}(y)\, s_\beta \varphi_{,1x}(x_1) - \\ - a_{ij11}(y)\, y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ij11}(y)\, V_{1,1x}(x_1)]\, (U^{(2)} - v^{(2)})_{i,jy} dy \geqslant 0\end{aligned} \tag{4.14}$$

for any $v \in \tilde{M}$.

We introduce the notation: $e = V_{1,1x}(x_1)$ is the axial deformation of the beam; $\rho_\alpha = u_{\alpha,1x1x}^{(0)}(x_1)$ is the curvature of the beam (in the Ox_α plane); $\psi = \varphi_{,1x}(x_1)$ is the angle of twist per unit length. Then mesh problem (4.14) can be rewritten in the form

$$\int_{P_1} [a_{ijkl} U^{(2)}_{k,ly} - a_{ij\beta 1}(y)\, s_\beta \psi - a_{ij11}(y)\, y_\alpha \rho_\alpha + a_{ij11}(y)\, e]\, (U^{(2)} - v^{(2)})_{i,jy} dy \geqslant 0 \tag{4.15}$$

for any $v \in \tilde{M}$.

In contrast to the linear case [3, 17], the resulting problem cannot be broken down into mesh problems corresponding to axial tension, bending, and torsion.

We will use $U^{(2)}(\psi, \rho_\alpha, e)$ to represent the solution of (4.15). It follows from (4.8) and (4.9) that

$$\sigma^{(-3)}_{ij} = a_{ijkl}(y)\, U^{(2)}_{k,ly}(\psi, \rho_\alpha, e) - a_{ij\beta 1}(y)\, s_\beta - \psi - a_{ij11}(y)\, y_\alpha \rho_\alpha + a_{ij11}(y)\, e. \tag{4.16}$$

Averaging (4.16) over a cell of the structure, multiplying by y_β, and averaging again, we obtain the constitutive relations of the given structure as a uniform unidimensional body:

$$\begin{aligned} \langle \sigma^{(-3)}_{11} \rangle &= \langle a_{11kl} U^{(2)}_{k,ly}(\psi, \rho_\alpha, e) \rangle - \langle a_{11\beta 1}(y) \rangle s_\beta \psi - \\ &- \langle a_{1111}(y)\, y_\alpha \rangle \rho_\alpha + \langle a_{1111}(y) \rangle e, \\ M^{(-3)}_{\beta i} &= \langle y_\beta a_{i1kl} U^{(2)}_{k,ly}(\psi, \rho_\alpha, e) \rangle - \langle y_\beta a_{i1\gamma 1}(y) \rangle s_\gamma \psi - \\ &- \langle y_\beta a_{i111}(y)\, y_\alpha \rangle \rho_\alpha + \langle y_\beta a_{i111}(y) \rangle e. \end{aligned} \tag{4.17}$$

5. Boundary Conditions. The nonlinearity of the problem does not affect boundary conditions (1.1) or expansion (2.1). Thus, the boundary conditions for the unidimensional problem are obtained in the same manner as in [3] and have the form

$$V_1(\pm 1) = 0, \quad u^{(0)}_\alpha(\pm 1) = u^{(0)}_{\alpha,1x}(\pm 1) = 0, \quad \psi(\pm 1) = 0. \tag{5.1}$$

6. Closed System of Equations for the Unidimensional Problem. As noted above, parts of Eqs. (3.3) and (3.5) turn out to be satisfied identically. We write the rest of these equations as:

$$M^{(-3)}_{\alpha i,1x} + \langle \sigma^{(-2)}_{i\alpha} \rangle = \langle g_i y_\alpha \rangle_\gamma + \langle f_i y_\alpha \rangle; \tag{6.1}$$

$$\langle \sigma^{(-2)}_{i1} \rangle_{,1x} = \langle g_i \rangle_\gamma + \langle f_i \rangle. \tag{6.2}$$

By differentiating (6.1) with $\alpha i = \alpha 1$ and using (6.2), we can exclude $\langle \sigma_{1\alpha}^{(-2)} \rangle$ from this equation when $i = 1$:

$$M^{(-3)}_{\alpha 1,1x1x} = -\langle g_\alpha \rangle_\gamma - \langle f_\alpha \rangle + \langle g_1 y_\alpha \rangle_{\gamma,1x} + \langle f_1 y_\alpha \rangle_{,1x}. \tag{6.3}$$

As can be seen, $\langle \sigma_{1\alpha}^{(-2)} \rangle$ corresponds physically to shearing forces [7]. When $i \neq 1$, we obtain torsion equations. Here, we introduce moment in torsion $M = M_{23}^{(-3)} - M_{32}^{(-3)}$ [3]. With allowance for the symmetry of $\sigma_{mn}^{(-2)}$ with respect to mn, we obtain the following from (6.1) for the turning moment

$$M^{(-3)}_{,1x} = \langle g_2 y_3 \rangle_\gamma - \langle g_3 y_2 \rangle_\gamma + \langle f_2 y_3 \rangle - \langle f_3 y_2 \rangle. \tag{6.4}$$

Together with constitutive relations (4.17) and boundary conditions (5.1), Eqs. (6.3), (6.4) form a closed system. Nonlinear constitutive relations (4.17) distinguish the present approach from conventional beam theories.

7. Modeling a Mesh Problem with Finite-Dimensional Problems. Mesh problem (4.14) can be examined relative to the displacements

$$U - y_\beta s_\beta e_\beta \psi - w^\alpha \rho_\alpha + y_1 e_1 e,$$

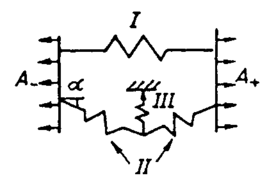

Fig. 3

where w^α are the displacements, determined by the condition $(\text{def } w^\alpha)_{kl} = y_\alpha \delta_{k1} \delta_{l1}$. Since the strains $y_\alpha \delta_{k1} \delta_{l1}$ satisfy the compatibility condition [7], the displacements w^α exist (and are determined in explicit form [7]). This observation is useful in introducing simplified models for mesh problems. As an example, we will present a simplified model of braided fibers (see Fig. 2) working in tension. The model is shown in Fig. 3, where I represents seven elastic elements corresponding to straight internal (woven) fibers, II corresponds to an elastic element which represents a braid, and III corresponds to a fiber−braid contact. The stiffness of the "included" contact is determined (see Fig. 2) similarly to the stiffness of the "floor" for beams on an elastic foundation [7]. The braid in Fig. 2 has six contact points within the cell. The stiffness of the contact can be determined on the basis of the fact that misaligned cylinders are in contact with one another in the given case. We should note that A_- and A_+ are subject to the condition that their mutual displacement be equal to e. We ignored the bending of the cell that might have occurred (due to its asymmetry).

8. Dynamic Problem. This problem arises when allowance is made for inertial forces [which corresponds to the replacement of $\varepsilon^{-2}\mathbf{f}$ by $\varepsilon^{-2}\mathbf{f} + \varepsilon^{-2}\rho \mathbf{u}_{,tt}^{\varepsilon}$ in Eq. (1.3)]. The analog of inequality (1.3) is obtained in this case (this is examined in greater detail in [5, 6]). The multiplier ε^{-2} in the dynamic term is connected with allowance for the order of the linear density of the beam. The formal asymptotic expansion which is analogous to (2.1)-(2.3) can be used to solve the given problem. As above, only the equation of transverse vibration is of interest in the calculations:

$$M^{(-3)}_{\alpha 1,1x1x} = -\langle g_\alpha \rangle_\gamma - \langle f_\alpha \rangle + \langle g_1 y_\alpha \rangle_{\gamma,1x} + \langle f_1 y_\alpha \rangle_{,1x} + \langle \rho \rangle u^{(0)}_{\alpha,tt} + \langle \rho y_\alpha \rangle u^{(0)}_{\alpha,tt1x}. \quad (8.1)$$

The nonclassical term in (8.1) is the term containing $u_{\alpha,tt1x}^{(0)}$. At the same time, the appearance of this term has a natural explanation within a mechanical context. The multiplier $\langle \rho y_\alpha \rangle$ with $u_{\alpha,tt1x}^{(0)}$ characterizes the asymmetry of the mass distribution over the cross section of the rod (this asymmetry being expressed in dynamic processes). For symmetrical beams, $\langle \rho y_\alpha \rangle = 0$.

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